

The multi-moment map of the nearly Kähler $S^3 \times S^3$

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Abstract

We describe the multi-moment map associated to an almost Hermitian manifold which admits an action of a torus by holomorphic isometries. We investigate in particular the case of a \mathbb{T}^3 action on the homogeneous nearly Kähler $S^3 \times S^3$. We find that the multi-moment map in this case acts more-or-less similarly to the moment map of a toric manifold, while the more general case does not.

1 Introduction

An almost Hermitian manifold (M, g, J) is *nearly Kähler* if $\nabla^g J$ is skew-symmetric. We say a nearly Kähler manifold is *strict* if it is not Kähler. The minimum dimension admitting strict nearly Kähler manifolds is 6, and there are only a handful of known examples of compact strictly Kähler 6-manifolds. The homogeneous spaces \mathbb{S}^6 , $\mathbb{S}^3 \times \mathbb{S}^3$, \mathbb{CP}^3 and $\mathrm{SU}_3/\mathbb{T}^2$ admit strict nearly Kähler structures, and there are no other homogeneous strict nearly Kähler 6-manifolds [1]. The only non-homogeneous examples that are known are cohomogeneity one structures on \mathbb{S}^6 and $\mathbb{S}^3 \times \mathbb{S}^3$ [3], and these are conjectured to be the only cohomogeneity one examples.

If one wants to look for higher cohomogeneity examples, one could look for strict nearly Kähler 6-manifolds admitting the action of a 3-torus \mathbb{T}^3 by holomorphic isometries. Of the list of known examples in the previous paragraph, only the homogeneous $\mathbb{S}^3 \times \mathbb{S}^3$ admits such a symmetry group. The purpose of this paper is to explore this example.

A compact Kähler 6-manifold admitting a \mathbb{T}^3 action of holomorphic isometries would be toric. Such a manifold could be studied with use of the moment map μ , which is a \mathbb{T}^3 -equivariant map from the manifold to the dual Lie algebra of the torus, \mathfrak{t}^* . Each fiber of μ is a \mathbb{T}^3 orbit, and the image of μ is the polyhedron which is the convex hull of the μ -image of the fixed points of the \mathbb{T}^3 action.

In general, an almost Hermitian 6-manifold admitting a \mathbb{T}^3 action of holomorphic isometries would not be toric. However, we can study the multi-moment

map ν associated to the 3-form $d\omega$. This is a \mathbb{T}^3 -equivariant map from the manifold to the three dimensional vector space $\Lambda^2\mathfrak{t}^*$, so one can hope that it will have similar properties to the momentum map of a toric 6-manifold. We find that multi-moment map of $\mathbb{S}^3 \times \mathbb{S}^3$ does have some similar properties and some differences with the momentum map of a toric 6-manifold, while a more generic almost Hermitian manifold can have a rather poorly behaved multi-moment map.

We find that the multi-moment map image $\Delta := \nu(\mathbb{S}^3 \times \mathbb{S}^3)$ of $\mathbb{S}^3 \times \mathbb{S}^3$ is convex and that its boundary $\partial\Delta$ contains the 1-skeleton of a regular tetrahedron. However, Δ bulges beyond the faces of the tetrahedron, and $\partial\Delta$ is smooth away from the vertices. Along $\partial\Delta$, each ν -fiber is a \mathbb{T}^3 orbit, but in the interior, each fiber contains two orbits. The following table compares the fiber types for the multi-moment map of $\mathbb{S}^3 \times \mathbb{S}^3$ to the moment map of a toric 3-manifold:

Fiber of a point in ...	μ toric 6-manifold	ν for nearly Kähler $S^3 \times S^3$
a vertex	{point}	\mathbb{T}^2
an edge	\mathbb{S}^1	\mathbb{T}^3
a face	\mathbb{T}^2	\mathbb{T}^3
the interior	\mathbb{T}^3	$\mathbb{T}^3 \amalg \mathbb{T}^3$

2 Torus actions on almost Hermitian structures

Let (M, g, J, ω) be an almost Hermitian manifold. Let \mathbb{T} be a torus acting on M by holomorphic isometries. Any vector $X \in \mathfrak{t}$ induces a vector field K_X on M , which is a holomorphic Killing vector field. This means that $\mathcal{L}_{K_X}g = 0 = \mathcal{L}_{K_X}J$. By the Leibniz rule, this implies that $\mathcal{L}_{K_X}\omega = 0$.

If (M, g, J, ω) is Kähler, so that ω is closed, then there exists a moment map $\mu : M \rightarrow \mathfrak{t}^*$ defined by

$$\langle d\mu, X \rangle = -K_X \lrcorner \omega,$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing of \mathfrak{t} and \mathfrak{t}^* .

If we do not require (M, g, J, ω) to be Kähler, there is a multi-moment map associated to the closed 3-form $d\omega$ [4]. This is the map $\nu : M \rightarrow \Lambda^2\mathfrak{t}^*$ defined by

$$\left\langle d\nu, \sum_i X_i \wedge Y_i \right\rangle = - \sum_i K_{X_i} \lrcorner (K_{Y_i} \lrcorner d\omega), \quad \forall \sum_i X_i \wedge Y_i \in \Lambda^2\mathfrak{t},$$

where here $\langle \cdot, \cdot \rangle$ is the natural pairing of $\Lambda^2\mathfrak{t}$ and $\Lambda^2\mathfrak{t}^*$. Recall that the Lie derivative acts on differential forms by

$$\mathcal{L}_X \tau = d(X \lrcorner \tau) + X \lrcorner d\tau.$$

We can use this to simplify our expression for the multimoment map ν :

$$\begin{aligned}
\left\langle d\nu, \sum_i X_i \wedge Y_i \right\rangle &= - \sum_i K_{X_i} \lrcorner (K_{Y_i} \lrcorner d\omega) = - \sum_i K_{X_i} \lrcorner (\mathcal{L}_{K_{Y_i}} \omega - d(K_{Y_i} \lrcorner \omega)) \\
&= \sum_i K_{X_i} \lrcorner d(K_{Y_i} \lrcorner \omega) = \sum_i \mathcal{L}_{K_{X_i}} (K_{Y_i} \lrcorner \omega) - d(K_{X_i} \lrcorner K_{Y_i} \lrcorner \omega) \\
&= \sum_i d\omega(K_{X_i}, K_{Y_i}).
\end{aligned}$$

Here we've used the fact that $\mathcal{L}_{K_X} \omega = 0 = [K_X, K_Y]$ and the Leibniz rule to get $\mathcal{L}_{K_{X_i}} (K_{Y_i} \lrcorner \omega) = 0$. This equation can be integrated to solve for ν :

$$\nu \left(\sum_i X_i \wedge Y_i \right) = \sum_i \omega(K_{X_i}, K_{Y_i}) + C$$

for some constant C . Note that we can always choose C to be 0, so we will.

Note that one cannot expect ν to behave well for an arbitrary Hermitian structure. Motivated by the behaviour of the moment map of toric manifolds, one could expect that ν is almost everywhere a submersion, which means that the (multi-)moment map locally separates orbits. The following proposition shows that this condition does not always hold:

Proposition 2.1. *Let (M, g, J) be an almost Hermitian manifold equipped with a torus \mathbb{T} acting by holomorphic isometries. Then there exists a metric \hat{g} related to g by a \mathbb{T} -invariant conformal factor such that the multimoment map $\hat{\nu}$ of $(M, \hat{g}, J, \mathbb{T})$ is not a submersion on some open set in M .*

Proof. If $\nu(M) = \{0\}$, then $\hat{g} = g$ satisfies the claimed property. Otherwise, there exists some $p_0 \in M$ with $\nu(p_0) \neq 0$. We can choose a smooth \mathbb{T} -invariant function ϕ so that $\phi(p) = -\log \|\nu(p)\|$ for all p in some neighbourhood U of p_0 . Consider the conformally related metric $\hat{g} = e^\phi g$. The multi-moment map with respect to the conformally related Kähler form $\hat{\omega} = e^\phi \omega$ is $\hat{\nu} = e^\phi \nu$. We chose ϕ so that $\hat{\nu}$ maps U into the unit sphere in $\Lambda^2 \mathfrak{t}^*$, so that $\hat{\nu}$ is not a submersion on U . \square

In the rest of the paper, we will describe the multi-moment map for a torus action on the homogeneous nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. We find that ν is a submersion near generic orbits, and show other similarities and differences to the moment map of a toric manifold.

3 Homogenous nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$

We begin by reviewing the definition of the homogenous nearly Kähler structure on $\mathbb{S}^3 \times \mathbb{S}^3$, following the work in [2].

We identify \mathbb{S}^3 with the unit sphere in the quaternions \mathbb{H} . For any $p \in \mathbb{S}^3$, $T_p \mathbb{S}^3 \subset T_p \mathbb{H}$ is the image of $T_1 \mathbb{S}^3$ by the pushforward of left-multiplication by p .

Identifying $T_p\mathbb{S}^3 \subset T_p\mathbb{H}$ with $p^\perp \subset \mathbb{H}$, this pushforward is simply quaternionic multiplication by p . Thus the basis $\{i, j, -k\}$ of $\text{Im } \mathbb{H}$ which is identified with $T_1\mathbb{S}^3$ gives a frame for $T_{(p,q)}\mathbb{S}^3 \times \mathbb{S}^3 = T_p\mathbb{S}^3 \oplus T_q\mathbb{S}^3$:

$$\begin{aligned} E_1(p, q) &= (pi, 0), & F_1(p, q) &= (0, qi), \\ E_2(p, q) &= (pj, 0), & F_2(p, q) &= (0, qj), \\ E_3(p, q) &= (-pk, 0), & F_3(p, q) &= (0, -qk), \end{aligned}$$

where i, j, k are imaginary quaternions satisfying $ij = k$.

The almost complex structure for the homogenous nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ is given in this frame by

$$J = \frac{1}{\sqrt{3}} \sum_{n=1}^3 (-E_n \otimes E^n + F_n \otimes F^n + 2F_n \otimes E^n - 2E_n \otimes F^n).$$

The metric g is given by the average of $g_{\mathbb{H}^2}$ and $g_{\mathbb{H}^2}(J\cdot, J\cdot)$, where

$$g_{\mathbb{H}^2} = \sum_{n=1}^3 \left((E^n)^2 + (F^n)^2 \right)$$

is the flat metric from \mathbb{H}^2 restricted to $\mathbb{S}^3 \times \mathbb{S}^3$. This gives

$$\begin{aligned} g &= \frac{4}{3} \sum_{n=1}^3 \left((E^n)^2 - E^n F^n + (F^n)^2 \right), \\ \omega &= \frac{4}{\sqrt{3}} \sum_{n=1}^3 E^n \wedge F^n. \end{aligned}$$

3.1 Torus actions

For unit quaternions $a, b, c \in \mathbb{S}^3$, the map

$$F_{a,b,c} : \mathbb{S}^3 \times \mathbb{S}^3 \rightarrow \mathbb{S}^3 \times \mathbb{S}^3 : (p, q) \mapsto (apc^{-1}, bqc^{-1})$$

is a holomorphic isometry [5].

Lemma 3.1. *The map*

$$F : \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \rightarrow \text{Aut}(\mathbb{S}^3 \times \mathbb{S}^3, J) \cap \text{Isom}(\mathbb{S}^3 \times \mathbb{S}^3, g) : (a, b, c) \mapsto F_{a,b,c}$$

is an injective homomorphism.

Proof. It is clear from the definition of F that $F_{a,b,c} \circ F_{a',b',c'} = F_{aa',bb',cc'}$, so that F is a homomorphism. To see that F is injective, let $F_{a,b,c} = \text{Id}$. Then

$$(1, 1) = F_{a,b,c}(1, 1) = (ac^{-1}, ab^{-1}),$$

so that $a = b = c$. For any $(p, q) \in \mathbb{S}^3 \times \mathbb{S}^3$,

$$(p, q) = F_{a,a,a}(p, q) = (apa^{-1}, aqa^{-1}).$$

Since this is true for all $p, q \in \mathbb{S}^3$, we find that a lies in the center of \mathbb{S}^3 . Since \mathbb{S}^3 has a trivial center, $a = 1$ as required. \square

Since the projection of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ onto any of its factors is a homomorphism, any abelian subgroup of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ must be a product of abelian subgroups of \mathbb{S}^3 . But every non-trivial abelian subgroup of \mathbb{S}^3 is of the form $\{e^{At}\}_{t \in \mathbb{R}}$ for some unit imaginary quaternion A . Thus, a maximal torus in $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ is of the form

$$\{(e^{At_1}, e^{Bt_2}, e^{Ct_3})\}_{(t_1, t_2, t_3) \in \mathbb{R}^3},$$

for some $A, B, C \in \mathbb{S}^2$, identifying \mathbb{S}^2 with the unit imaginary quaternions. A routine computation shows that the image of such a torus under F is generated by the Killing vector fields

$$\begin{aligned} K_1 &= (Ap, 0), \\ K_2 &= (0, Bq), \\ K_3 &= (-pC, -qC). \end{aligned}$$

Since $p \in \mathbb{S}^3$, $Ap = p\bar{p}Ap$, so that these Killing vector fields can be written in terms of the frame $(E_1, E_2, E_3, F_1, F_2, F_3)$ as

$$\begin{aligned} K_1 &= ((\bar{p}Ap) \cdot i)E_1 + ((\bar{p}Ap) \cdot j)E_2 - ((\bar{p}Ap) \cdot k)E_3, \\ K_2 &= ((\bar{q}Bq) \cdot i)F_1 + ((\bar{q}Bq) \cdot j)F_2 - ((\bar{q}Bq) \cdot k)F_3, \\ K_3 &= (C \cdot i)(E_1 + F_1) + (C \cdot j)(E_2 + F_2) - (C \cdot k)(E_3 + F_3), \end{aligned}$$

where \cdot is the dot product on \mathbb{H} . This allows us to compute

$$\begin{aligned} \frac{\sqrt{3}}{4}\omega(K_1, K_2) &= (\bar{p}Ap) \cdot (\bar{q}Bq), \\ \frac{\sqrt{3}}{4}\omega(K_1, K_3) &= (\bar{p}Ap) \cdot C, \\ \frac{\sqrt{3}}{4}\omega(K_2, K_3) &= (\bar{q}Bq) \cdot C. \end{aligned}$$

Choosing the basis $\left\{ \frac{\sqrt{3}}{4}(K_n \wedge K_m)^* \right\}_{1 \leq n < m \leq 3}$ for $\Lambda^2 \mathfrak{t}^*$, this allows us to write the multi-moment map as

$$\nu(p, q) = ((\bar{p}Ap) \cdot (\bar{q}Bq), (\bar{p}Ap) \cdot C, (\bar{q}Bq) \cdot C).$$

Note that $\nu^{-1}(0)$ is the union of the Lagrangian torus orbits. The example of a Lagrangian torus in [2] can be found with the values $A = B = i$ and $C = j$.

3.2 Behaviour of the multi-moment map

We will first describe the image of the multi-moment map ν . Then we will describe the structure of its fibers.

For $X \in \mathbb{S}^2$, let us define a map

$$\pi_X : \mathbb{S}^3 \rightarrow \mathbb{S}^2 : p \mapsto \bar{p}Xp.$$

When $X = i$, this is the usual Hopf fibration. For general X , π_X also identifies \mathbb{S}^3 as a \mathbb{S}^1 bundle over \mathbb{S}^2 .

Define a function

$$\bar{\nu} : \mathbb{S}^2 \times \mathbb{S}^2 \rightarrow \Lambda^2 \mathfrak{t}^* : (x, y) \mapsto (x \cdot y, x \cdot C, y \cdot C),$$

so that $\nu = \bar{\nu} \circ (\pi_A \times \pi_B)$. Let $\Delta = \nu(\mathbb{S}^3 \times \mathbb{S}^3) = \bar{\nu}(\mathbb{S}^2 \times \mathbb{S}^2)$ with interior $\mathring{\Delta}$.

Lemma 3.2. $\Delta = \left\{ (X, Y, Z) : X \in [f_-(Y, Z), f_+(Y, Z)] \right\}$, where

$$f_{\pm}(Y, Z) = YZ \pm \sqrt{1 - Y^2} \sqrt{1 - Z^2}.$$

Proof. Let $C^\perp \leq \text{Im } \mathbb{H}$ be the plane orthogonal to C . Use the orthogonal decomposition $\text{Im } \mathbb{H} = \mathbb{R}C \oplus C^\perp$ to write any $(x, y) \in \mathbb{S}^2 \times \mathbb{S}^2$ as

$$x = (x \cdot C)C + x^\perp, \quad y = (y \cdot C)C + y^\perp, \quad x^\perp, y^\perp \in C^\perp.$$

Then we have the following relations:

$$\begin{aligned} 1 &= x \cdot x = (x \cdot C)^2 + \|x^\perp\|^2, \\ 1 &= y \cdot y = (y \cdot C)^2 + \|y^\perp\|^2, \\ x \cdot y &= (x \cdot C)(y \cdot C) + x^\perp \cdot y^\perp. \end{aligned}$$

If $\bar{\nu}(x, y) = (X, Y, Z)$, then

$$X = x \cdot y = YZ + x^\perp \cdot y^\perp.$$

By the Cauchy-Schwarz inequality, $|x^\perp \cdot y^\perp| \leq \|x^\perp\| \|y^\perp\| = \sqrt{1 - X^2} \sqrt{1 - Y^2}$, so that $f_-(Y, Z) \leq X \leq f_+(Y, Z)$. It is clear that by varying x and y , any value of X in this range can be attained, proving the claimed result. \square

Lemma 3.3. Δ is convex.

Proof. By the previous lemma, it suffices to prove that $\mp f_{\pm}$ is a convex function. This follows from the computation

$$\det \circ \text{Hess}(f_{\pm}) = \left(\frac{X\sqrt{1 - Y^2} + Y\sqrt{1 - X^2}}{\sqrt{1 - X^2}\sqrt{1 - Y^2}} \right)^2 \geq 0.$$

\square

Proposition 3.4. $\partial\Delta$ is contained in the affine variety $0 = F(X, Y, Z) = 2XYZ - X^2 - Y^2 - Z^2 + 1$. The set of singular points of $\partial\Delta$ is

$$V := \{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}.$$

Proof. By lemma 3.2, $\partial\Delta$ is given by points $(X, Y, Z) \in \mathbb{R}^3$ with $X = f_{\pm}(Y, Z)$. Such a points satisfy the relation $X^2 = 2XYZ + (1 - Y^2)(1 - Z^2) - Y^2Z^2$, which can be rearranged to form $F(X, Y, Z) = 0$.

The singular points of $\partial\Delta$ are the points where F and ∇F both vanish. The set of points where ∇F vanishes are $\{(0, 0, 0)\} \cup V$. The result follows since $F(0, 0, 0) = 1 \neq 0$, while F vanishes on V . \square

Proposition 3.5. *The line segment between any two points in V lies in $\partial\Delta$.*

Proof. We will show that the line segment between $(1, 1, 1)$ and $(1, -1, -1)$ lies in $\partial\Delta$, with the other line segments following similarly. This line segment is parametrized by

$$\gamma : [-1, 1] \rightarrow \mathbb{R} : t \mapsto (1, t, t).$$

Consider the functions $\tilde{f}_{\pm}(X, Y, Z) := f_{\pm}(Y, Z) - X$. By lemma 3.2,

$$\partial\Delta = \left\{ \vec{X} \in \mathbb{R} : \tilde{f}_+(\vec{X}) = 0 \geq \tilde{f}_-(\vec{X}) \text{ or } \tilde{f}_+(\vec{X}) \geq 0 = \tilde{f}_-(\vec{X}) \right\}.$$

We compute

$$\tilde{f}_{\pm} \circ \gamma(t) = t^2 - 1 \pm (1 - t^2).$$

Thus for $t \in [-1, 1]$, $\tilde{f}_+ \circ \gamma(t) = 0$ and $\tilde{f}_- \circ \gamma(t) = 2(t^2 - 1) \leq 0$. This shows that $\gamma(t) \in \partial\Delta$ as required. \square

By the previous proposition, we find that $\partial\Delta$ contains the 1-skeleton of the regular tetrahedron with vertices V . However, the full tetrahedron is properly contained in Δ . In Figure 1, we see that Δ is a regular tetrahedron with convexly bulging sides:

Proposition 3.6. *$\bar{\nu}$ has three different orbit types according to the following table:*

$\dim \text{Span}\{x, y, C\}$	location on Δ	$\bar{\nu}^{-1}(\bar{\nu}(x, y))$	$\nu^{-1}(\bar{\nu}(x, y))$
1	V	$\{(x, y)\}$	\mathbb{T}^2
2	$\partial\Delta \setminus V$	\mathbb{S}^1	\mathbb{T}^3
3	$\overset{\circ}{\Delta}$	$\mathbb{S}^1 \amalg \mathbb{S}^1$	$\mathbb{T}^3 \amalg \mathbb{T}^3$

Proof. Let $(x, y) \in \mathbb{S}^2 \times \mathbb{S}^2$ such that $\dim \text{Span}\{x, y, C\} \neq 1$. Then one of x or y is not $\pm C$. We will treat the case when $x \notin \{\pm C\}$, with the other case following similarly.

Write $\bar{\nu}(x, y) = \tau = (\tau_1, \tau_2, \tau_3)$. Let $(x_0, y_0) \in \bar{\nu}^{-1}(\bar{\nu}(x, y))$. Thus $\tau_2 = x_0 \cdot C \notin \{\pm 1\}$. This relation defines a circle S_0 on \mathbb{S}^2 of possible x values. For a fixed $x_0 \in S_0$, the relations $\tau_1 = x_0 \cdot y_0$ and $\tau_3 = y_0 \cdot C$ define two circles S_1 and S_2 on \mathbb{S}^2 centered at x_0 and C respectively, which intersect at possible solutions for y_0 . Two circles can intersect in at most 2 points. If S_1 and S_2 do not intersect, then $\bar{\nu}^{-1}(\tau) = \emptyset$, contradicting $\tau \in \Delta$. If S_1 and S_2 intersect at exactly one point y_0 , then y_0 is a linear combination of the centers x_0 and C of S_1 and S_2 . If they intersect at two points, then each intersection point is not a

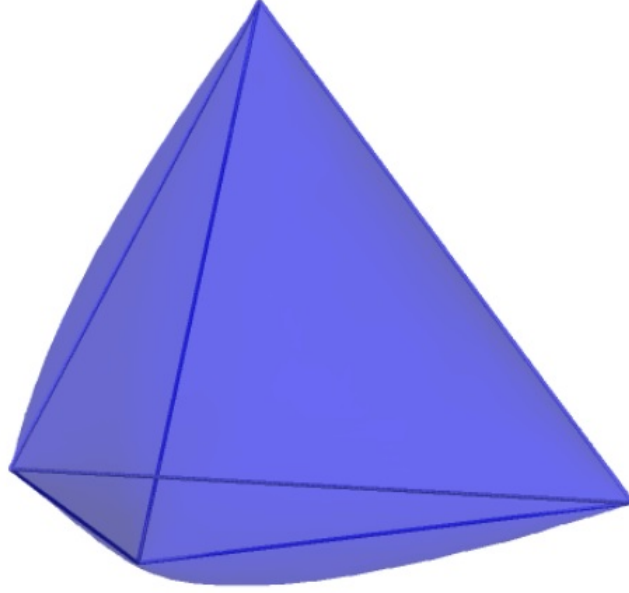


Figure 1: The multi-moment map image of $\mathbb{S}^3 \times \mathbb{S}^3$

linear combination of x_0 and C . Since there is a circle worth of choices for x_0 , this gives the last two rows of the table.

The remaining points in $\mathbb{S}^2 \times \mathbb{S}^2$ satisfy $\dim \text{Span}\{x, y, C\} = 1$. This is equivalent to $x, y \in \{\pm C\}$, which define 4 points. To see that these points live in different $\bar{\nu}$ fibres, the following table evaluates $\bar{\nu}$ at each of these points:

x	y	$\bar{\nu}(x, y)$
C	C	$(1, 1, 1)$
C	$-C$	$(-1, 1, -1)$
$-C$	C	$(-1, -1, 1)$
$-C$	$-C$	$(1, -1, -1)$

Thus the singleton fibres get mapped to V . We've established the correspondence between the first and third rows in the claimed table. The last column follows since $\nu = \bar{\nu} \circ (\pi_A \times \pi_B)$, where $\pi_A \times \pi_B$ determined a \mathbb{T}^2 bundle. The second column follows from the description in lemma 3.2, noting that $\partial\Delta$ consists of the points where the Cauchy-Schwarz inequality is an equality, which are the points where $\{x, y, C\}$ are linearly dependent vectors in $\text{Im } \mathbb{H}$. \square

Note that $\bar{\nu}^{-1}(\mathring{\Delta})$ has two connected components determined by the sign of $\det\{x, y, C\}$, while $\bar{\nu}^{-1}(\partial\Delta)$ is the vanishing locus of $\det\{x, y, C\}$. It follows that ν is a submersion along $\nu^{-1}(\mathring{\Delta})$.

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References

- [1] Jean-Baptiste Butruille. Homogeneous nearly Kähler manifolds. *Handbook of pseudo-Riemannian geometry and supersymmetry*, pages 399–423, 2010.
- [2] Bart Dioos, Luc Vrancken, and Xianfeng Wang. Lagrangian submanifolds in the nearly Kaehler $S^3 \times S^3$. *arXiv preprint arXiv:1604.05060*, 2016.
- [3] Lorenzo Foscolo and Mark Haskins. New G_2 -holonomy cones and exotic nearly kähler structures on S^6 and $S^3 \times S^3$. *Annals of Mathematics*, 185(1):59–130, 2017.
- [4] Thomas Bruun Madsen and Andrew Swann. Closed forms and multi-moment maps. *Geometriae Dedicata*, 165(1):25–52, 2013.
- [5] Fabio Podestà and Andrea Spiro. Six-dimensional nearly Kähler manifolds of cohomogeneity one. *Journal of Geometry and Physics*, 60(2):156–164, 2010.